Chapter

A Modern Review of Wavelet Transform in Its Spectral Analysis

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Abstract

The spectral analysis, in much aspects as are the wavelet transform in its numerous versions and its relation with other transforms and special functions requires a special review, since the exploration in the frequency domain to the wavelet transform is more detailed and majorly more specific in different applications. For example, the wavelet transform of special function can be very useful to create and design special signal filters or, for example, to the interphase between reception-emission devices with sensorial parts of the human body. Also the quantum wavelet transform is very useful in the spectral study of traces of particles. Likewise, in this chapter, these aspects are considered as an inherent property of the wavelet transform in the spectral exploration of some phenomena. Finally, general results to the discrete case are given, which is analyzed to the wavelet transform and its spectra.

Keywords: discrete Fourier transform, discrete wavelet transform, fast Fourier transform, Gabor transform, short-time Fourier transform, spectra, quantum wavelet transform, wavelet transform

1. Introduction

Likewise, we consider a set of functions

$$\psi_{1k}, \psi_{2k}, \dots, \psi_{\mathrm{nk}}, \dots \in L^2(R), \tag{1}$$

which define a Hilbert basis of square integrable functions [1]. Likewise, for each $j, k \in Z$, the ψ_{jk} represents dyadics and dilations of ψ, given by the functions:

$$\psi_{\mathrm{jk}}(x) = 2^{\frac{j}{2}}\psi\left(2^j x - k\right), \tag{2}$$

$\forall j, k \in Z$. Likewise, for a function $\xi(x) \in L^2(R)$, and using the orthonormal functions family, by completeness we have:

$$\xi(x) = \sum_{-\infty}^{+\infty} c_{\mathrm{jk}}\psi_{\mathrm{jk}}(x), \tag{3}$$

Then the convergence of the series will be understood to be convergence in norm. Likewise, a representation of $\xi(x)$ is known as a wavelet series with wavelet coefficients c_{jk}. This implies that an orthonormal wavelet is self-dual. Then the wavelet integral transform is the integral transform [2] given for[1]:

$$W_{\psi}\xi = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{+\infty}\bar{\psi}\left(\frac{x-b}{a}\right)\xi(x)\mathrm{dx}, \tag{4}$$

Here $a = 2^{-j}$ is the binary dilation or dyadic dilation, and $b = k2^{-j}$ is the binary or dyadic position. Then the wavelet transform can be modified depending on the response treatments that are given. For example in the images compression through impulse function $x(n) = \delta(n - n_i)$, for a discrete signal where impulse response can be used to evaluate the image compression-reconstruction system, the wavelet transform has been modified.

As has been said in different signal treatments, one of fundamental problems in electronics is obtaining a sufficiently clean signal in the different processes of communication, perception, and management of the signals in different continuous and discrete domains. For example, in signal processing in accelerometers for gait analysis, where actually is necessary to implement a good programming in real time for drones or other devices of vehicles, even human body parts with accelerations process in fault detections for design of low power pacemakers and also in ultrawideband in wireless [3]; the cleaning of signal is fundamental.

The wavelet transforms as transformation should allow only changes in time extension, but not shape. This could be affected by choosing suitable basis functions that allow for this. For example, in numerical analysis, we can consider the scale factor $c_n = c_0^n$, with the discrete frequency $\eta_m = \mathrm{mLc}_0^n$, having the wavelets (considering the discrete formula with the basis wavelet):

$$\Psi(k,n,m,) = \frac{1}{\sqrt{c_0^n}}\Psi\left[\frac{k-\mathrm{mc}_0^n}{c_0^n}L\right] = \frac{1}{\sqrt{c_0^n}}\Psi\left[\left(\frac{k}{c_0^n}-m\right)L\right], \tag{5}$$

where such discrete wavelets can be used through the discrete wavelet transform version:

$$W_D(n,m) = \frac{1}{\sqrt{c_0^n}}\sum_{k=0}^{K-1}w(k)\Psi\left[\left(\frac{k}{c_0^n}-m\right)L\right], \tag{6}$$

whose continuous (or analogic) is the standard wavelet transform:

$$W(c,\eta) = \frac{1}{\sqrt{c}}\int_{-\infty}^{+\infty}w(t)\Psi\left(\frac{t-\eta}{c}\right)\mathrm{dt}, \tag{7}$$

where c, is a scaling factor, and η, represents time shift factor. For the case (7) the Fourier transformation of signal $w(k)$, is computed with the FFT. An adequate selection of a discrete scaling factor c_n will be necessary. Changes in the time extension are expected to conform to the corresponding analysis frequency of the basis function, based on the uncertainty principle of signal processing.

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[1] To recover the original signal $w(t)$, the first inverse continuous wavelet transform can be used: $\xi(\mathrm{t}) = \chi_{\psi}^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{W}_{\psi}\xi(\mathrm{a},\mathrm{b})\frac{1}{\sqrt{|\mathrm{a}|}}\tilde{\psi}\left(\frac{\mathrm{t}-\mathrm{b}}{\mathrm{a}}\right)\mathrm{db}\frac{\mathrm{da}}{\mathrm{a}^2}$.

For example, the wavelet transform of Shannon function can be very useful to creation of windows $\Psi^{\text{Sha}}(\omega)$, through functions $\text{Sha}(t)$, and with gate functions $\Pi(x)$, useful in the signal analysis by ideal band-pass filters that define a decomposition known as Shannon wavelets. Also, for example, the complex-valued Morlet wavelet is closely related to human perception, both hearing and the vision [4].

Likewise, the transition for "classical" wavelet transform (with some modifications accepted) to quantum wavelet transform can be approached by factoring the classical operators for the transformation into direct sums, direct products, and dot products of unitary matrices. Likewise, the permutation matrices play a vital role [5].

2. From the signal resolution problems until biological-sensorial perception

A fundamental property of the wavelet transform and the signal resolution problem can be discussed and explored simultaneously in time and frequency domains starting from the wavelet spectra:

$$W(\omega) = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{+\infty} w\left(\frac{t-b}{a}\right)e^{-j\omega t}\text{dt} = \sqrt{|a|}W(a\omega)e^{-j\omega t}, \tag{8}$$

where $W(\omega)$ is the Fourier transform of the basic wavelet $w(t)$. If the wavelets are normalized in terms of amplitude, the Fourier transforms of the wavelets with different scales will have the same amplitude that is suitable for implementation of the continuous wavelet transform using the frequency domain filtering. This property is fundamental in the samples of frequency pulses of signal spectra, since it shows that a dilatation t/a $(a > 1)$ of a function in the time domain produces a contraction $a\omega$, of its Fourier transform, which are "spectral wavelets" corresponding. Likewise, the term t/a has a metrology of frequency, which is equivalent to ω. However, in the technical convention, this term is known as scale, since the term "frequency" is reserved for the Fourier transform. Then the design of signal filters in frequency obey to the correlation between the signal and the wavelets, in the time domain that can be written as the inverse Fourier transform of the product of the conjugate Fourier transforms of the wavelets and the Fourier transform of the signal:

$$W_{\xi}(a,b) = \frac{\sqrt{|a|}}{2\pi}\int_{-\infty}^{+\infty} \Xi(\omega)W(j\omega a)e^{-j\omega t}d\omega \tag{9}$$

The Fourier transforms of the wavelets are referred to as the wavelet transform filters. The impulse response of the wavelet transform filter $\sqrt{|a|}W(a\omega)$ is the scaled wavelet $\frac{1}{\sqrt{|a|}}w\left(\frac{t}{a}\right)$. Therefore, the wavelet transform is a collection of wavelet transform filters with different scales, a. Then we can relate the short-time Fourier transform (STFT) [6] with the idea of the wavelets to determine the sinusoidal frequency and phase content of local sections of a signal considering as changes over time. If we introduce the Gaussian function, which can be regarded as a window function, then the STFT is the Gabor transform. Here the Gabor atoms or functions used to build from translations and modulations of generating function a family of functions are constructed and characterized.

Likewise, we can have direct applications of the STFT, to samples in real time of the complex processes, which require a speed compute of data through direct relation between machine and real-time domains in the measured and perception of the phenomena. Likewise, the STFT is performed on a computer using the fast Fourier transform (FFT), so both variables are discrete and quantized.

Secondly the Morlet transform is a Gabor transform consisting of a wavelet composed of a complex exponential (carrier) multiplied by a Gaussian window (envelope). This wavelet is closely related to human perception, both hearing [2] and vision. Then the functions related with these bio-sensorial perceptions use Sha(t) functions as special Gabor functions to discriminate steps of signal spectra in the perception and create of a signal response audible or visible required to the eye organ, the iris of eye, in the case of the vision and to audition, we have the audiphones that amplify the sounds to equilibrate the lack of the eardrum or other parts of middle and inner ear to perceive the sounds.

3. Some important results in discrete signal analysis

Let S^Nbe the complex sphere of dimension N, and let

$$\ldots, e^{-2\Omega j}, e^{-\Omega j}, 1, e^{\Omega j}, e^{-2\Omega j}, \ldots, \tag{10}$$

be a linear basis of signals space in $L^2[K]$ that generates the subspace W, such that $\forall w \in W$ is

$$w = \sum_{n=1}^{N} c_n e^{-n\Omega j}, \tag{11}$$

We define the space of nilpotent classes on $E_{[K]}$, (being $E_{[K]} = E_1 \oplus \ldots \oplus E_n$, [7] the total discrete signal space) with the component:

$$N(E_{[K]}) = \{F \in D'(G^0/K) | F = 0\}, \tag{12}$$

Proposition 3.1 [8, 9]. If $z \in N(E_{[K]})$, and if $\beta \in H^i(n_0, L^2[K])$, (then)

$$z\beta = e^{-n\Omega j} x[m] = p(z) x[n], \tag{13}$$

where $x[n]$ is a Gabor discrete function[2].

Proof. Here

$$n_0 \simeq n, \quad n_0 \simeq p_I,$$

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[2] A discrete version of Gabor representation is

$x(t) = \sum_{n=-\infty}^{\infty} \sum_{m=\infty}^{\infty} C_{nm} g_{nm}(t)$,

with $g_{nm}(t) = s(t - m\eta_0) e^{-n\Omega jt}$.Similar to the DFT (discrete Fourier transformation) we have:

$x(k) = \sum_{n=-\infty}^{\infty} \sum_{m=\infty}^{\infty} C_{nm} g_{nm}(k)$,

where the Gabor basis functions are $g_{nm}(k) = s(k - m\eta_0) e^{-n\Omega jk}$.

where p_I is the unitary sphere $p \cap g_I$, where g_I, $= [g, g]$. We demonstrate on the dimension i, of the cohomological space $H^i(g_I, L^2[K])$. If $i = n = \dim n_0$, then

$$H^i(g_I, L^2[K]) = n_0 * \otimes L^2 / n_0 L^2, \tag{14}$$

Therefore, z acts by $I \otimes p(z)$. Then $p(z)$ acts for $(C \otimes \pi)p(z)$.(Then)

$$(I \otimes p(z))\beta = p_C(z)\beta, \tag{15}$$

As a special note, we have as a particular example an LTI system $L(e^{-n\Omega j}) = H(\Omega)e^{-n\Omega j}$, where $H(\Omega)$ is a projection of the system function.

Likewise, the result to $i = r + 1 \leq n$, then demonstrate for $i = r$. Let C be the periodic complex. Let $\alpha \in H * (X_i, Q_b Z_a)$, such that $\alpha(g \otimes v)gv$. Then α is the homeomorphism

$$\alpha : \mathrm{Hom}_K(p_I, C) \to \mathrm{Hom}_K(p_I, L^2[K]), \tag{16}$$

Let $X = \ker\alpha$, where specifically

$$X = \{\alpha \in \mathrm{Hom}_K(C, L^2[K]) | \alpha(g \otimes v) = 0, \quad \forall g \in U(g), v \in L^2[k]\}, \tag{17}$$

Then we have

$$0 \to X_i \to C \to L^2 \to 0, \tag{18}$$

Now $U(g)$, is a $U(p_I)-$complex free under left translations. Therefore

$$H^i(p_I, C) = 0, \tag{19}$$

$$\forall C = E_{[K]} \otimes Q_b Z_a, \text{(then)}$$

$$H^i(p_I, E_{[K]} \otimes Q_b Z_a) = 0, \tag{20}$$

$\forall j < n$, and $b \equiv a \bmod j$. Then the long exact sequence of discrete cohomology for this case of periodic complexes and $N(E_{[K]})-$complex is:

$$0 \to H^i(p_I, L^2[K]) \to H^{i+1}(p_I, X) \to H^{i+1}(p_I, E_{[K]} \otimes Q_b Z_a) \to 0, \tag{21}$$

where such injection implies the result.

A study realized in signal and systems analysis on a linear system can be approximated in the time-frequency domain due to the composition of an analysis filter-bank, a transfer matrix (sub-band model) and a synthesis filter-bank, which is a method known as sub-band technique.

In the varying case, time-frequency representations of LTV systems have connection with the Gabor expansion of signals through the corresponding integral equation. Then we will have an integral equation with Gabor function. For example, a work realized in that sense is the creation of 3D Gabor frame based in spatial spectral integral equation designed to solve the scattering from dielectric objects embedded in a multilayer medium. Likewise, this is based on the Gabor frame, as a new set of basic functions (belonging to a basis) [10] together with a set of equidistant Dirac-delta test functions.

Proposition 3.2. Exists an isomorphism given for the DFT that maps the proper nilpotent classes of the system controlled under transformations of p_I. Then DS-TFT is the FFT.

Let DFT be the isomorphism of the discrete signals:

$$E_{[K]} \rightarrow \tilde{E}_{[K]}, \tag{22}$$

where the explicit rule for any $\forall v \in L^2[K]$ is

$$\mathrm{DFT}(v) = (1/N)\mathrm{DFT}(v) = \mathrm{DWT}(v), \tag{23}$$

Then in each component of the space $E_{[K]}$, $(E_{[K]} = E_1 \oplus \dots \oplus E_n.)$ we have:

$$F^{(k)} = 0, \tag{24}$$

where the DS-TFT satisfies in short-time interval. In each component, we have:

$$U(p_I)\mathrm{Hom}_K(F, L^2[K]) = \chi_\Lambda p_\xi, \tag{25}$$

which exists $\mathrm{FFT}^\sim(v)$, such that

$$\mathrm{DWT}(v) = \mathrm{FFT}(v), \tag{26}$$

More details of the demonstration can be consulted in [11].

4. Conclusions

In this introductory chapter, the various and several advantages of the wavelet transform and its properties on the signal and system analysis have been shown, considering different specialized window functions and the wavelet function basis. Likewise, wavelet analysis is known for its successful approach to solving the problem of signal analysis in both the time domain and frequency domain. Also, the analysis of the nonstationary signal generated by physical phenomena has a great challenge for various conversion techniques. In several studies, it has been shown that the transformation techniques such as Fourier transform and short Fourier transform fail to analyze nonstationary signals. But wavelet transform methods may be able to efficiently analyze both stable and unstable signals. All this the author develops with precision and accuracy. In the Gabor transform, the resolution analysis considers the uncertainty principles on nilpotent Lie groups and their corresponding algebras, which were established in the propositions given through spectral analysis given in the classes $H^i(p_I, L^2[K]), H^{i+1}(p_I, X)$, and $H^{i+1}(p_I, E_{[K]} \otimes Q_b Z_a)$. A scheme with neural network as components of a dynamical system can be proposed to demonstrate that using neural networks and linear filters in cascade and/or feedback configurations, a rich class of models of signaling and systematization in wide perspective and prospective can be constructed, considering the different filters designed by the different wavelet transform versions in short-time resolution or conventional resolution improving the canonical Fourier transform resolution. The multi-resolution analysis or multi-scale approximation can design a method considering practically the relevant discrete wavelet transforms (DWT), which can be considered as a fundamental set of special functions to realize approximations to solutions of different processes in time and the justification for the algorithm of the fast wavelet transform (FWT), for the calculating methods develop started with good wavelet bases.

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Nomenclature

STFT	Short-Time Fourier Transform
FFT	Fast Fourier Transform
DFT	Discrete Fourier Transform
LTV	Linear-Time Varying System
$DS-TFT$	Discrete Short-Time Fourier Transform
$w(t)$	Basic wavelet
$W(\omega)$	Fourier transform of the basic wavelet $w(t)$
ψ_{jk}	Dyadic and dilations of the wavelets
$x[n]$	Discrete signal. In the proposition 2.1, is a discrete Gabor function
$E_{[K]}$	Discrete signal space. This space is a Hilbert space on the discrete domain K. Its orthogonal decomposing is $E_{[K]} = E_1 \oplus \dots \oplus E_n$. In the case of wavelets, the components $E_j (j = 1, 2, \dots, n)$ are dyadic translations and dilations of wavelet $w(k)$
DWT	Discrete Wavelet Transform

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