Chapter

Introductory Chapter: Frontiers and Future Developments of the Complex Analysis

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1. Introduction

The complex analysis development can establish three fundamental ways of modern research with the goal to solve certain conjectures and complete certain theories as the Lie groups representation theory, obtaining of a general cohomology for certain integral operators in the differential equations solution and extend these solutions to the meromorphic context [1].

The three ways are defined for the relations between the complex analysis, with the cohomology with coefficients in holomorphic vector bundles, the hyper-complex analysis in a quaternionic algebra and the unitary representations in the Lie group theory [2].

Likewise, some objects as modular forms can be the key for the multidisciplinary investigations in complex functions without the condition that $f(z)$, be holomorphic in the upper half-plane. Then the modular forms are meromorphic functions [3]. The before, can be reflected in the satisfying of the certain functional equation $\mathrm{Tf}(z) = \xi$, in which the conditions of the complex function $f(z)$, respect to the transformation T, can be relaxed and the corresponding modular groups can be smaller groups. Such is the case, for example for the functional equation of the homographic transformation.

The transcendence goes more beyond of the analyticity problems in complex analysis. The study of the complex Riemannian manifolds results relevant in algebraic geometry and strings theories, where complex Riemann surfaces as complex submanifolds can be useful in the determination of integration invariants. Also in the Fröbenius distribution for the determination of integral submanifolds as cycles, whose values are co-cycles in a spectrum of the category Spec_k, of the category of the complex vector spaces Vec_C.

The theory of Riemann surfaces can be applied to the space $G \setminus H*$, to obtain further information about modular forms and functions. Likewise, the modular and cusp forms spaces $M_k(G)$, and $S_k(G)$, for a group G, are of finite dimension, and [4] their dimensions can be computed thanks to the Riemann-Roch theorem in terms of the geometry of the G$-$action on H. For example to the Lie group $SL(2,\mathbb{Z})$, this finite dimension is computed as[1]:

$$\dim_C M_k(\mathrm{SL}(2,Z)) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor, & k \equiv 2(\mathrm{mod} 12) \\ \left\lfloor \frac{k}{12} \right\rfloor + 1, & \text{else} \end{cases} \tag{1}$$

[1] Here $\lfloor , \rfloor$, is a floor function.

In projective spaces, the functions are homogeneous polynomials. These algebro-geometric objects are the sections of a sheaf (one could also say a line bundle in this case). Precisely here arises the re-interpretation of the differential operators through the homogeneous polynomials of the corresponding lines bundles sections of their respective connections [5] (**Figure 1**).

However, there is other conjecture established years ago, which establish the sentence.

Conjecture 1.1. All complex function (analytic function) $f(z)$, can be determined for hypercomplex functions [6].

This could be obtained in a clear way, without pass for the holomorphicity.

This represents all a joint research program between the hypercomplex analysis and the integral operators cohomology, and for it is established determine an integral operators theory that establish equivalences between cohomology classes on different complex base spaces and the cycles and co-cycles of these under the corresponding integral operators.

Likewise, can be obtained integral representations that are realizations of certain unitary representations of Lie groups such as $U(n), \mathrm{SU}(2), \mathrm{SU}(n), \mathrm{SU}(2,2), \ \mathrm{SU}(p,q)$, and $U(p,q)$, for the obtaining of the general solution of a partial differential equations in contexts of the quaternion algebra. And not only that, also are contemplated the equivalences of the corresponding cohomology classes obtained through the restricted objects to a space of certain dimension, for example, the hypercomplex analysis in $\mathbb{R}^4 \cong \mathbb{C}^2$, and in $\mathbb{C}^4$, is a quaternion analysis. Other example we can find in the integrals of the integral geometry of $G_{2,4}(\mathbb{C})$, through the space $\mathbb{P}^3(\mathbb{C})$, which establishes a twistor geometry [2, 7].

However, the isomorphism relations in homogeneous vector bundles are equivalent to the followed in cohomology developed in vector tomography. Likewise, the intertwining integral operators between cohomological classes of both contexts (respective cycles) result be cohomological classes in contexts of lines bundles (or equivalent co-cycles). In this sense a conjecture that can be proved is: “the Penrose transform is a vector Radon transform on homogeneous vector bundles sections in a Riemannian manifold” And considering the re-construction of a Riemannian manifold through cycles, we can consider the following conjecture also.

Conjecture 1.2. [2] The twistor transform is the generalization of the Radon transform on lines bundles and the Penrose transform is a specialization of the twistor transform in S^4.

Of fact, the twistor transform can be determined for the Penrose transforms pair on $G-$orbits of projective spaces $\mathbb{P}^-$, and $\mathbb{P}^+$. Then here arises the mentioned before in modular forms, the use of the lines bundles restricted to the polynomial rings corresponding to the subjacent operators on the complex vector spaces. In a more general sense, the vector bundles which refer us, are the seated in the homogeneous spaces $G^{\mathbb{C}}/L$, with homogeneous subspaces $G^{\mathbb{C}}/Q$, where is subjacent a

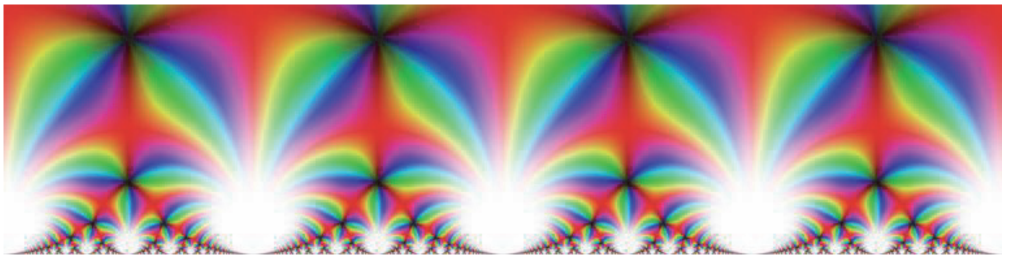

Figure 1.
The j$-$*invariant on the complex interval* $[-2, 2] + [0, 1]i$. *It supports working with modular transformations and evaluating theta sums. With this construction of blocks, one can easily implement other modular forms such as Eisenstein series. (*http://fredrikj.net/blog/2014/10/modular-forms-in-arb/*).*

holomorphic manifold with $G-$ orbits (parabolic orbits) of $G^{\mathbb{C}}/L$, which have an induced Haar measure in each closed submanifold given for the flags of the corresponding holomorphic vector $G-$ bundle. Then their orbital integrals are calculated on said orbits, obtaining invariants applicable to whole the homogeneous space in question, $G^{\mathbb{C}}/L$. This process called of orbitalization of the homogeneous space $G^{\mathbb{C}}/L$, is based in major part in the inclusion of homogeneous spaces obtained for reduction of $G^{\mathbb{C}}/L$, through its holonomy [7].

The spaces or cohomology groups of the complex corresponding of sheaves $H^s(G/H, \nu_n)$, result be fine representations of $G^{\mathbb{C}}/L$, through of the flags bundle corresponding to the complex holomorphic $G-$ bundle in $G^{\mathbb{C}}/L$. The realizations of these unitary representations are the orbital integrals on the flags that are the $K-$ orbits of the corresponding vector $G-$ bundle. This is precisely used in representations of $G-$ (a complex Lie group) constructed on spaces of holomorphic sections of vector bundles and generalizations as the studied for the modular forms. The problem that here arise (in the unitary representations) is the verifying of the Hermitian forms calculated through the diverse integral realizations with the corresponding representations choosing the smooth and minimal globalizing, adequate. For it is used in a general version of the complex cohomology known as $\overline{\partial}-$ cohomology called Serre generalization [7].

However, this is not reduced to the obtaining of representations of complex groups. Also the obtaining of an analytic function $f(z)$, of a space to other is realized accord to the integral cohomology $(n-1-q)-\overline{\partial}-$ cohomology $H^{n-1,q}(\mathbb{C}^n/D, V)$, where D, is a linearly concave domain on $\mathbb{C}^2$. Likewise, for example, if we consider the space $\mathbb{C}^2$, the integral operator that obtains a harmonic function in $\mathbb{R}^4$, through calculated functions of $C^\infty(\mathbb{C}^3)$, on lines S^1, in $\mathbb{C}^2$, comes given for [8, 9]:

$$\varphi(w,x,y,z) = \int_{S^1} f[(w+x)+(y+\mathrm{i}z)\zeta, (y-\mathrm{i}z)+(w-x)\zeta, \zeta]d\zeta, \tag{2}$$

which represents a transformation in cycles (lines) of a flag manifold $\mathbb{F} = (L|L \subset \mathbb{R}^4)$. Likewise complex functions in certain complex manifolds can be obtained through complex or hypercomplex functions defined in complex subspaces (or submanifolds) which could be varieties in an algebraic context, for certain conditions of its cohomology. For example in a complex sheaves cohomology, results interesting for problems of differential equations whose operators are germs of these sheaves.

Other study is the singularities study through varieties considering newly the complex varieties, in a pure algebraic treatment, or using the analyticity as a concept required for contour integrals to define a singularity as a pole of an analytic function.

Likewise in concrete, through an analytic functions study, can be extended and generalized the Cauchy integral as through integrals of contour can be generalized functional in a cohomology of contours (cohomological functional). The Morera's and Cauchy-Goursat's theorems can be applied in a sense geometrical more general as cohomological functional of contours whose residue value can be modulo $l(S^1) = 2\pi$. For example, two singularities or poles. This could represent the surface of the real part of the function $g(z) = \frac{z^2}{z^2+2z+2}$. The moduli space of these points are less than 2 and thus lie inside one contour. Likewise, the contour integral can be split into two smaller integrals using the Cauchy-Goursat theorem having finally the contour integral [10] $\oint_C g(z)\mathrm{d}z = \oint_C \left(1 - \frac{1}{z-z_1} - \frac{1}{z-z_2}\right)\mathrm{d}z = 0 - 2\pi\mathrm{i} - 2\pi i = -4\pi i$. This is a good example of traditional cohomological functional element of $\mathrm{H}^f(\Pi - \ell', \Omega^r) = \mathbb{C}$.

As a way to extend the meromorphic functions, the Nevanlinna theory searches generalizations of extensions of analytic functions to algebroid functions, holomorphic curves [10], holomorphic maps between complex manifolds of arbitrary dimension, quasi-regular mappings and minimal surfaces [11]. This is being applicable in moduli problems and Shimura varieties.

2. Future research

Finally, the future researches in complex analysis will be centered on major research of meromorphicity, in the automorphicity [12, 13], bi-holomorphicity, Riemann-Zeta function, Hodge theory and the complex ideals for improve methods of homogeneous polynomials in the complex vector bundles studied in complex Riemaniann manifolds and their submanifolds, involving singularities as poles or insolated singularities of rational curves. The automorphic forms generalize the Cauchy and Morera's theorems in an arithmetic sense inside the analytic functions theory and their probable holomorphicity. The automorphic forms are the other basic operations that arise inside the mathematics in the modular forms.

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