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Introductory Chapter: Frontier Research on Integral Equations and Recent Results

Francisco Bulnes

"Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country"

- David Hilbert

1. General discussion

The themes of recent research are focused on nonlinear integral equations [1], the new numerical and adaptive methods of resolution of integral equations [2], the generalization of Fredholm integral equations [3] of second kind, integral equations in time scales and the spectral densities [3, 4], operator theories for nonsymmetric and symmetric kernels [1, 5], extension problems to Banach algebras to kernels of integral equations [5–7], singular integral equations [10], special treatments to solve Fredholm integral equations of first and second kinds, nondegenerate kernels [3, 6] and symbols of integral equations [7], topological methods for the resolution of integral equations and representation problems of operators of integral equations.

Now, well, the field of the integral equations is not finished yet, not much less with the integral equations for which the Fredholm theorem is worth [fredholm], nor with the completely continuous operators, since there exist other integral equations developed of the Hilbert theory respect to the Fredholm discussion, and studies on singular integral equations, also by Hilbert, Wiener and others [8]. Arise numerical and approximate methods on the big vastness that give the Banach algebras, even using probabilistic measures to solve some integral equations in the ambit of distributions and stochastic process. Likewise, there arise integral equations in which the proper values are corresponded to linearly independent infinite proper functions. Such is the case, for example, of the Lalesco-Picard integral equation:

$$\omega(t) - \lambda \int_{-\infty}^{+\infty} e^{-|t-s|} \omega(s) \mathrm{d}s = f(t), \tag{1}$$

in which the kernel $e^{-|t-s|}$ is not of L^2 class and gives a continuous spectra, or even, we consider nonlinear integral equations, etc., that represent the last and recent studies on integral equations after of their study considering extensions of the Banach algebras to integral operators that can define to this proposit, for example, to singular integral equations.

Likewise, as special case, for their important theory, we can treat the singular integral equations of Cauchy. This theory was created almost immediately after the Fredholm theory, and their beginning is given in the "Lecons de Mécanique Céleste" by Poincaré and Fichot [9], and to the Hilbert works on contour and boundary problems of the analytic functions theory.

A possible treatment, bringing the Cauchy ideas together with Banach algebras, is the consideration of the Calkin algebra $\frac{B(X)}{K(X)}$, on a Banach space X, likewise as the operators of subalgebras of this special Banach algebra (e.g., the algebra of the bounded operators $B(X)$, and $K(X)$, the ideal of compact operators) [10]. For example, consider the bounded operators in a Banach space with closed range and with kernel and co-kernel of finite dimension. These are called Fredholm operators and are the operators that give invertible elements of the Calkin algebra. The operators of the Calkin algebra radical are called Reisz operators and can be characterized spectrally and in terms of the dimensions of $\mathrm{Rec}\left((\lambda I - T)^k\right)$, $\ker\left((\lambda I - T)^k\right)$, etc. Very questions on these algebras are motive of modern research. However, also the integral equations research has developed more the functional analysis, considering the function theory, integral transforms and the Kernels study in a wide form.

For other side, a general resolution method to the singular integral equations cannot be given in detail on the effective resolution of these equations, because is followed the research on a general methods to this integral equations class through certain special functions and integral transforms, which are of diverse and varied nature [5, 11]. In fact, the resolution of singular integrals considering the Hilbert transform and the Fourier transform [11] has been in the last years strongly researched. Here we only consider the intimate relation between this singular integral equations theory with the analytic functions theory and special functions related with the regularity and completeness of the solutions required.

One of the new developments on nonlinear integral equations are followed to the Hammerstein integral equations [12], which is written as

$$\omega(t) + \int_a^b K(t,s)f(s,\omega(s))\mathrm{d}s = 0, \quad a \leq t \leq b \tag{2}$$

where $K(t,s)$ and $f(t,s)$ are given functions, while $\omega(t)$ is the unknown function. Hammerstein considered for $K(t,s)$, a symmetric and positive Fredholm kernel. This last condition establishes that all their eigenvalues are positive. Thus, the function $f(t,s)$ is continuous and satisfies $|f(t,s)| \leq C_1|s| + C_2$, where C_1 and C_2 are positive constants and C_1 is smaller than the first eigenvalue of the kernel $K(t,s)$; then, the Hammerstein integral equation has at least one continuous solution. Also are considered certain observations on the no decreasing of the function $f(t,s)$, on s, considering fix t, from the interval (a,b). The Hammerstein's equation cannot have more than one solution. This property holds also if $f(t,s)$ satisfies the condition

$$|f(t,s_1) - f(t,s_2)| \leq C|s_1 - s_2|, \tag{3}$$

where the positive constant C is smaller than the first eigenvalue of the kernel $K(t,s)$. A solution of the Hammerstein equation may be constructed by the method of successive approximation. In regard to this point, many approximation methods are designed to solve these integral equations and other nonlinear integral equations. Also of interest are the recent developments on Hammerstein-Volterra integral equations:

$$f(t) = \omega(t) + \int_0^t K(t,s)f(s,\omega(s))\mathrm{d}s, \quad 0 \leq t \leq 1 \tag{4}$$

In the aspect of the linear integral equations has been important the study of the Volterra integral equations on time scale, where have more importance the initial

value problems with unbounded domains. Likewise, the development on the alternate form of a linear integral equation is given as:

$$f(t) = \omega(t) + \int_a^t B(t,s)\omega^\sigma(s)\mathrm{ds}, \qquad t \in I_T \tag{5}$$

where $B(t,s)$ is a kernel that comes of a Banach algebra, and ω^σ, arises naturally of changing dynamics problems, for example the economic dynamics. Some aspects in their prospective can be extended to the nonlinear case.

Other studies go on to develop generalizations of integral equations of Fredholm type using Weyl fractional integral operators and the kernel as product of certain generalized functions of special functions such as the H functions and the I functions. This establishes new techniques in function theory and functional analysis relating some integral transforms such as the Mellin transform [13].

Other developments start the probabilistic methods searching the obtaining of a solution of some integral equations of the second kind and Volterra integral equation, thinking in stochastic phenomena where is necessary determine an aleatory behavior.

Author details

Francisco Bulnes[1,2]

1 Research Department in Mathematics and Engineering, TESCHA, Mexico

2 IINAMEI, Mexico

*Address all correspondence to: francisco.bulnes@tesch.edu.mx

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